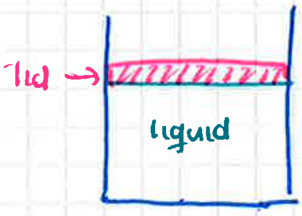
Lecture 2. Fluid statics

Fluid mechanics is concerned with the study of fluids that are deformed by flow, and as a result the field is often called fluid dynamics. But even in the absence of motion, fluids have remarkable characteristics. When they don't move, fluids are said to be at equilibrium and their study is that of fluid statics. One of the most important properties of fluids at equilibrium is their ability to sustain and exert pressure.

Pressure

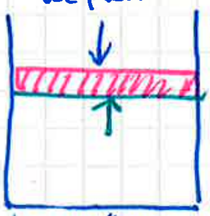
Let's imagine the following experiment.



We have a container with a liquid and a lid

we push on the lid

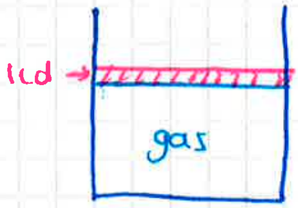
e.g. put a weight on it



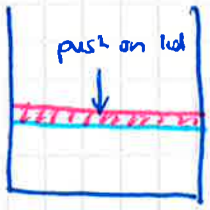
no visible change → incompressible

liquid "pushes back" with the same magnitud in the opposite direction so the liquid doesn't move.

The ability of the liquid at rest to generate a force in response to an external push is known as pressure.



push on lid



gas decreases volume → compressible

gas will compress ... (decrease its volume)

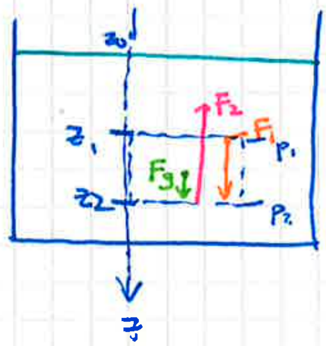
There is less volume for particles in container → more collisions between them → pressure in container increases

At some point increase in gas pressure that force applied by gas will balance external force

Hydrostatic pressure.

When a fluid is at rest, the pressure at any point does not depend on direction. However the pressure does vary with depth.

To see how this works imagine submerging yourself in water. The deeper you go, the more water lies above you, and the greater the pressure you feel. We can calculate this in pressure with depth by considering a small "cube" of fluid and balancing the forces acting on it.



- Consider the air pressure at sea level $(z = z_0 = 0)$ is $p = p_0$
  $p(z_0) = p_0$
- We take z to increase in the downward direction to measure depth
- We'll consider a volume of water that extends from z_1 to z_2
- The forces acting in this volume of water are
  $F_g \equiv$ gravitational force due to its mass
  $F_2 \equiv$ net force upward from water below (normal force)
  $F_1 \equiv$ downward force due to everything above

The fluid is at rest so the sum of the forces on it must be zero:

$F_g + F_2 + F_1 = 0$ where $F_g = mg = \rho V g$ where ρ = mass density; V = volume
$F_2 = p_2 A$ where $p_2 = p(z_2)$
$F_1 = p_1 A$ where $p_1 = p(z_1)$

note that F_2 is in the opposite direction to F_g & F_1:

$\rho V g - p_2 A + p_1 A = 0$ dividing by A (remember $V = A\Delta z$)

$\Rightarrow p_2 - p_1 = \rho g \Delta z$ where Δz is the height of our volume.

so an increase in depth Δz causes an increase in pressure $\Delta p = \rho g \Delta z$

setting z=0 at the surface of the water, we can see that the pressure at depth z will be:

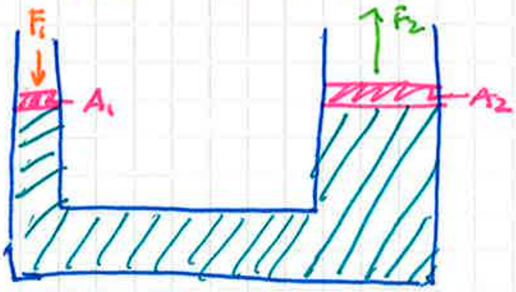
$\boxed{p = p_0 + \rho g z}$ hydrostatic pressure

Pascal's principle

As discussed above we can consider most liquids to be incompressible, being incompressible has an immediate consequence known as Pascal's principle.

Pascal's principle: A change in pressure applied to an enclosed incompressible fluid is transmitted instantaneously and undiminished to every part of the fluid and to its container.

In everyday life you encounter this principle at work when squeezing a tube of mustard or toothpaste. This is also the principle used in the design of the hydraulic lever, a tool used to lift heavy objects using only a small force.



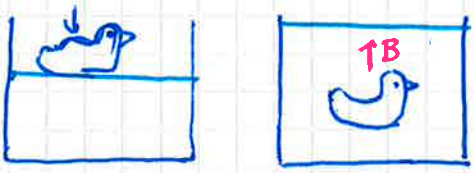
When we apply a downward force F_1 to the small area A_1, the pressure in the fluid is increased by: $p = \frac{F_1}{A_1}$

this creates an upward force
$F_2 = pA_2 = F_1 \left(\frac{A_2}{A_1}\right)$ on the output side

so ~~resultant~~ input force is increased by the ratio of the areas of the two pistons.

Archimede's principle and buoyancy force

The fact that pressure in a fluid increases with depth leads to the concept of buoyancy.
let's think of the following setting:



Take a rubber duck initially located outside the liquid and try to submerge it in the fluid.
To keep it submerged one needs to apply a downward force to it. why?

This is because once its submerged in the fluid, the duck is subject to the same distribution of hydrostatic pressure as if it were not there.

The fluid pressure continues to increase with depth and therefore the pressure pushing the duck down is lower than the pressure pushing back up resulting in a net force. The force due to the displaced fluid is called buoyancy and its magnitude is equal to the weight of the fluid displaced by the rubber duck in this case.

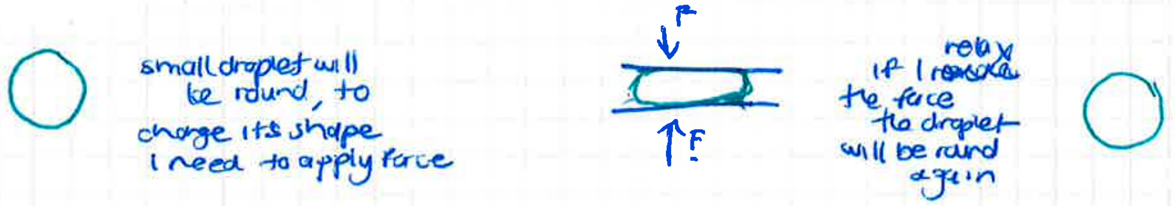
Thus we arrive at Archimedes's principle: the buoyancy force on an object equals the gravitational force on an equal size volume of water and is directed upward.

$$\boxed{F_B = \rho_{water} V g \hat{z}}$$

Archimedes's principle also applies to partially submerged objects (floating) if we only count the volume of the submerged part.

Surface tension

Another important property of liquids at equilibrium is their cohesion. Consider a droplet:



In other words the droplet has a property to want to remain round this cohesive property is called surface tension ~~also~~ and it has a molecular origin. Particles in the liquid interact with their neighbours through long-range attractive forces. It is therefore energetically favorable for a given particle to have as many neighbours as possible to decrease the overall internal molecular energy in the liquid.

It is therefore energetically unfavorable for the ~~liquid~~ fluid to have too many particles on its surface and the lowest energy state is obtained when the fluid has the smallest surface area possible. (to have as few ~~pa~~ surface particles as possible).

Therefore we see there is an energetic cost associated with increasing the surface area of the liquid. In order to increase the surface area it's necessary to provide external energy, this is why we need external forces to squeeze the droplet.

The surface tension of a liquid measures the energy of the liquid per unit of surface area. So we can measure it in Joules/m²

If surface tension would be the only energy to consider drops ~~with~~ would always be spherical as they are in space!

Surface tension can also be defined as the force per unit length along any line of the liquid's surface. (It is measured in N/m)

Just so you get an idea of the surface tension of various liquids at room temp:

	ethanol	acetone	glycerol	water	Hg
$\gamma \left[\frac{mN}{m}\right]$	23	24	63	73	435

Laplace pressure

Another way to lower the surface energy of a droplet is to simply reduce the radius of the droplet. To do so we need to compress the liquid inside raising its pressure. The energy needed to do this is given by pressure x volume.

The total energy of a drop of radius R is given by:

$$E = E_{surface} + E_{volume} = \underbrace{4\pi R^2 \gamma}_{\text{this term wants to reduce the radius of drop}} + \underbrace{\frac{4}{3}\pi R^3 \Delta p}_{\text{this term doesn't want to reduce it}}$$

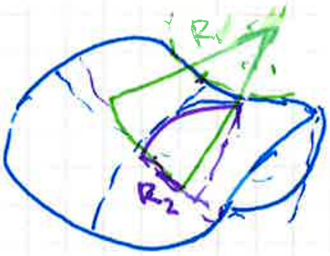
where Δp is the difference in pressure inside & outside of the drop.

these 2 terms "compete" until the drop achieves a radius that balances them. To find when this happens we derive the equation above with respect to R and set it equal to zero:

$$\frac{\partial E}{\partial R} = 8\pi R \gamma + 4\pi R^2 \Delta p = 0 \Rightarrow \boxed{\Delta p = \frac{2\gamma}{R}}$$ where $\Delta p = p_{in} - p_{out}$

Laplace pressure = pressure that a curved interface exerts on the inner fluid.

With this in mind we can generalize this to calculate the ~~pressure~~ Laplace pressure due to a boundary between 2 fluids with arbitrary shape:



$$\Delta P = \gamma \cdot (\text{curvature of surface}) = \gamma\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

where R_1, R_2 are the radii of curvature characterizing the surface

Wetting

when a fluid drop is put in contact with a rigid surface an additional source of energy comes into play, namely that due to the interactions between fluid and surface particles.

Fluid particles might favor interactions with other fluid particles or they might favor interactions with surface particles over other fluid particles.

complete wetting

~~fluid spread~~ droplet spreads completely on a surface

e.g water on ultraclean glass

partial wetting

fluid droplet spreads partially on a surface (takes the form of a partial sphere)

$\theta \equiv$ contact angle

angle between the liquid surface and the rigid wall (measured inside the liquid)

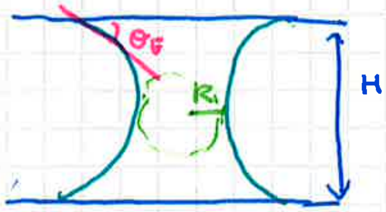
$\theta < \frac{\pi}{2}$ → e.g water on glass
→ hydrophilic surface
→ droplet is less than half of a sphere

$\theta > \frac{\pi}{2}$ → e.g water on grass
→ hydrophobic surface
→ drop is more than half a sphere

Consequences of surface tension

Example 1. Capilary adhesion

consider 2 glass plates and a liquid drop between the plates. We'll assume that the liquid wets the glass



we can estimate the adhesion between the 2 plates due to the liquid using calculating the Laplace pressure

$H \equiv$ distance between plates
$\theta_E \equiv$ wetting angle (we'll use it to estimate one of the rad of curvature)
R rad of curvature at center of droplet.

Using the definition of laplace pressure for this case:

$$\Delta p = \gamma\left(\frac{1}{R} - \frac{\cos\theta_E}{(H/2)}\right)$$ in the case $H \ll R$ (plates are very close)

$$\Rightarrow \Delta p \simeq \frac{2\gamma\cos\theta_E}{H}$$ How strong is this force?

$$F = \Delta p\, A = \left(\frac{2\gamma\cos\theta_E}{H}\right)A$$

consider the following estimates,
$\theta_E \simeq 0$
$H = 10\,\mu m$ size of gap
$A = \pi R^2$ where R is the radius of the droplet $R = 1\,cm$
$\Rightarrow A = \pi(1\times10^{-4})\,m^2$

$\gamma = 0.072\ N/m$ surface tension of water

putting the numbers in:

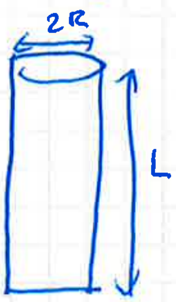
$$F \simeq \frac{2(0.072\,N/m)}{10\times10^{-6}\,m}\pi\times10^{-4}\,m^2$$

$$\simeq 0.072\times10^2\,N = 7.2\,N$$; quite a strong force!

Example 2. Surface tension induced stabilities

Rayleigh-Plateau instability

The fact that a liquid has surface energy leads to the famous Rayleigh-Plateau instability, wherein long fluid threads cannot remain elongated but break up into small droplets. This means we can blame surface tension for our dripping taps!



consider an elongated cylinder of fluid of length L and radius R

Volume $V = \pi R^2 L$

surface area $A = 2\pi R(L+R) \simeq 2\pi RL = \frac{2V}{R}$

for a long cylinder

Now let's "cut" the cylinder into N identical droplets each of radius a

By conservation of volume $N\left(\frac{4\pi a^3}{3}\right) = V$ so the number of droplets we have is:

$N = \frac{3V}{4\pi a^3}$ Now, the total surface area of the droplets is:

$\tilde{A} = N(4\pi a^2) = \frac{3V}{a}$

Then the ratio of the two surface areas is:

$$\frac{\tilde{A}}{A} = \frac{\frac{3V}{a}}{\frac{2V}{R}} = \frac{3R}{2a}$$ so for $a > \frac{3}{2}R$ the droplets will have a surface area smaller than the cylinder

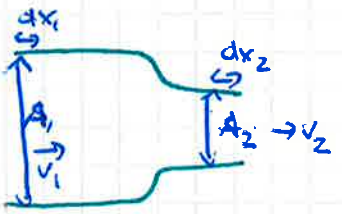
∴ it's energetically more favorable for an elongated cylinder of fluid to break into droplets if they are large enough.

Where have you seen this?
every time you open the tap!

Now we're ready to get started w/ some fluid dynamics.

Volume flow rate

Consider an incompressible fluid flowing through a round pipe with varying radius



Fluid incompressible ⇒ amount of fluid that goes into the pipe on the left = amount of fluid that exits the pipe on the right

$$A_1 dx_1 = A_2 dx_2$$

at a time dt : $A_1 v_1 dt = A_2 v_2 dt$

$$A_1 v_1 = A_2 v_2 \equiv \text{volume flow rate} = R_v$$

cross sectional area × fluid speed

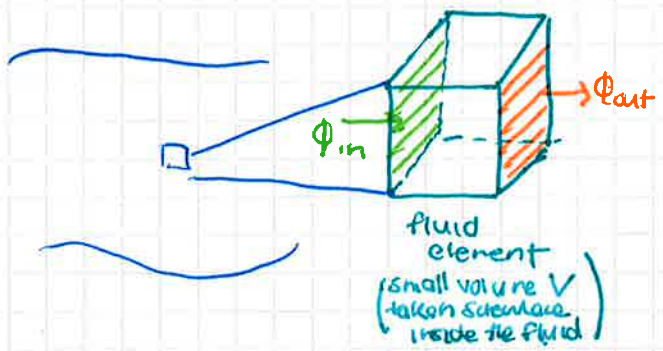
$[R_v]$ = volume/time

Because for an incompressible fluid the volume flow rate is conserved, the speed of the fluid is inversely proportional to the cross-sectional area of the pipe. From the definition of R_v we can also define the mass flow rate by multiplying R_v by ρ (fluid's mass density).

$R_m = \rho A v$ mass flow rate, which is conserved as well.

The continuity equation

We have looked at a very specific configuration, an incompressible fluid in a cylindrical pipe, but we can use similar arguments to derive a very useful property of fluid flows: the continuity equation.



Since fluid element has volume V, its mass is given by:

$$m = \int \rho \, dV = \rho V$$

if ρ is constant

Now we consider the flux of fluid into volume V. For any area bounding the volume the flux is defined as:

$$\phi = \rho \underline{v} \cdot d\underline{A}$$

where $d\underline{A}$ is a vector w/ magnitude that scales w/ the size of the area and direction ⊥ to surface.

$$= \rho v \, dA$$

$v \perp$ to dA

this is the mass flow rate from the previous section, but for an arbitrary volume w/ an arbitrary boundary.

Now the net flux will be the flux summed over all pieces of surface area dA, so we have the following surface integral:

$$\phi_{net} = \oint \rho \underline{v} \cdot d\underline{A} \quad (*)$$

Also:
A positive net flux corresponds to a net decrease in mass of the volume element per unit time so:

$$\phi_{net} = -\frac{\partial m}{\partial t} = -\frac{\partial}{\partial t}\int \rho \, dV = -\int \frac{\partial \rho}{\partial t} dV \quad (**)$$

So now we have that the surface integral (*) equals the volume integral (**):

$$\underbrace{\oint \rho \underline{v} \cdot d\underline{A}} = -\int \frac{\partial \rho}{\partial t} dV \quad \Rightarrow \quad \int \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{v}) \right] dV = 0$$

we'll switch this surface $\int$ for a volume $\int$ using the divergence theorem
$\int_V \nabla \cdot (\rho \underline{v}) \, dV$

since this applies to any volume V
⇒ integrand must be zero

$$\therefore \boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{v}) = 0 \quad \text{continuity equation}}$$

general statement of mass conservation
allows for changes in fluid density in time and space.

In the case the density is constant in time and space (⇒ incompressible fluid) this simplifies to:

$\nabla \cdot \underline{v} = 0$ continuity equation for incompressible fluid

Stream function
For incompressible flows, the continuity equation gives us a relation between components of the fluid velocity $\underline{v}$. In 2 dimensions this relation allows us to describe all properties of a flow using a scalar function

For a 2D incompressible fluid:

$v(x,y)$ $\quad \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \quad \Rightarrow \quad v_x = \frac{\partial \psi}{\partial y}(x,y)$

$$v_y = -\frac{\partial \psi}{\partial x}(x,y)$$

$\psi(x,y) \equiv$ stream function
lines along which ψ is constant are called streamlines

A streamline is a line that is everywhere tangent to the local flow direction.

If we have a steady flow (i.e. time independent) the streamlines are constant in time and coincide w/ the path a tracer particle takes.

Bernoulli's equation
We'll use conservation of energy to derive Bernoulli's equation for incompressible, steady flow.
consider an arbitrary flow tube:

on one end of the tube fluid is (xsectional area A_1) moving at vel v_1 and the pressure is P_1

at the other end of the tube (xsectional area A_2) the fluid moves at vel v_2 and the pressure is P_2

we want to know at which speed v_2 the fluid leaves the tube;
if $v_2 \neq v_1$ the kinetic energy changes by

$$\Delta K = \frac{1}{2} \rho V (v_2^2 - v_1^2)$$

if ΔK decreases ⇒ fluid did work
if ΔK increases ⇒ something did work on the fluid
↓
in this case either a difference in the pressure or gravity

If there is a height difference Δh, as sketched, the work done by gravity on the fluid is:

$$W_g = -\rho V g h$$

gravity points down so an increase in height means that work has been done against the direction of gravity.

If the pressure at point 2 differs from the pressure at point 1, the amount of work done by the pressure gradient is:

$$W_p = -(p_2 - p_1)V$$

Now equating the amount of work to the change in kinetic energy:

$$-\rho V g h - (p_2 - p_1)V = \frac{1}{2}\rho V(v_2^2 - v_1^2)$$

$$\Rightarrow \boxed{p_1 + \frac{1}{2}\rho v_1^2 + \rho g z_1 = p_2 + \frac{1}{2}\rho v_2^2 + \rho g z_2}$$ where $h = z_2 - z_1$ and z_i is the z coord of point i

Since there is nothing special about points 1 and 2, the equation must hold for any two points in the tube so we have:

$$p + \frac{1}{2}\rho v^2 + \rho g h = \text{constant}$$ Bernoulli's equation

If $h=0$ we can write Bernoulli's law for two points as:

$$p_2 - p_1 = \frac{1}{2}\rho v_2^2 - \frac{1}{2}\rho v_1^2$$ this requires no dissipation (we know that for most fluids this is not true)

The level of dissipation changes dramatically depending on the pipe configuration

- flow through a contraction → reasonable estimate (flow 'squeezes', acceleration is smooth, little loss)
- flow through an expansion → bad estimate
- flow through an abrupt expansion → terrible estimate

flow expands, eddies / turbulence, energy loss,